Notes on free fall of a particle and bouncing on a reflecting surface

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(Submitted March 31, 1997)

Considerable progress has recently been made in controling the motion of free atomic particles by means of light pressure exerted by laser radiation. The free fall of atoms and bouncing on a reflecting surface made from evanescent wave formed by internal reflection of a quasiresonant laser beam at a curved glass surface in the presence of homogeneous gravitational field has been observed. In this paper we present the energy quantization of this system by making use the asymptotic expansion method. It is shown that for large n the levels go like $n^{2/3}$ which may be compared with n^2 for the infinite square well.

PACS number(s): 03.65.Bz

Considerable progress has recently been made in our understanding of atomic environtment behaviour, even which allowed us to control the atom. Atoms have been held near stationary [1], thrown upwards without heating [2, 3], made to produce quantum interference after macroscopic path separations [4], and trapped in quantum wells [5, 6]. Such improvements are mainly accelerated by the invention of a new mechanics called wave mechanics or quantum mechanics. Although this new mechanics is different with the classical mechanics, there are many systems however, showed similar properties both in classical and quantum mechanical. One example is the case of simple harmonic oscillator which shows several agreement in average energy calculation. As shown in many standard textbooks, the average kinetic and potential energies of this system are the same in the two mechanics, i.e. $\frac{1}{2}E_0$. One could find that this remains true for the excited levels.

Instead of reviewed the problems on harmonic oscillator, let us now consider the problem of free fall of a particle in homogeneous gravitational field and bouncing on an elastically reflecting surface. This problem ussualy found in classical, for example, a ball in basketball games. This mechanism found possible to exist in the atomic scale. Many experiments already performed have shown that multiple bouncing of atoms on a surface can be established [7, 8]. The experiment is performed by measuring the average of an ensemble of independent atoms. It is known that such systems involved a damping and the (classical) sources of damping have been understood [9]. The quantum damping and the connection with an experiment already performed in the classical regime, however, is studied in [10].

In recent paper we focused on the technique of deriving the energy quantization of this system. Let us introduce the potential energy of the form

$$V(z) = \begin{cases} mgz, & (z > 0) \\ \infty, & (z \le 0) \end{cases} \tag{1}$$

which corresponds to the situation of free fall of a particle and bouncing condition, where m is mass of the particle, g is the acceleration due to gravity and z is the vertical height above the surface. The Schrödinger equation for positive z and energy eigenvalue E is then

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dz^2} + mgz\psi = E\psi. \tag{2}$$

The boundary conditions of the problem suggest that $\psi = 0$ for $z \le 0$. By introducing the caracteristic length of the system,

$$z_0 = (\frac{\hbar}{2m^2 g})^{1/3}. \tag{3}$$

it is shown in [11] that the solutions can be expressed in term of Airy equation, take the form [10]

$$\phi_n(z) = \mathrm{C}_n \mathrm{Ai}(\frac{z}{z_0} - \lambda_n). \tag{4}$$

with C_n's are normalization constants and $-\lambda_n$ is the nth zero of Airy function, related to the coresponding energy eigenvalue

$$E_n = \frac{\hbar^2}{2mz_0^2}\lambda_n. \tag{5}$$

To see the way let us putting $z = \xi z_0 + \eta$, where z_0 is as described in Eq. (3) and $\eta = E/mg$. By doing a little algebra to this identity we can obtain Eq. (5) and substitution in Eq. (2) we have then the form

$$\frac{d^2\psi}{d\xi^2} - \xi\psi = 0. \tag{6}$$

which is a form of Bessel's equation. One series solution of the modified Bessel equation

$$\frac{d^2 I}{dx^2} + \frac{1}{x}\frac{dI}{dx} - (1 + \frac{p^2}{x^2})I = 0. \tag{7}$$

is found to be [12]

$$I_p(x) = (\frac{1}{2}x)^p \sum_0^\infty \frac{(\frac{1}{4}x^2)^s}{s!(s+p)!}. \tag{8}$$

In our problem the solutions are Bessel function of order 1/3, more commonly known as Airy function [11] and denoted by $\mathrm{Ai}(\xi)$ and $\mathrm{Bi}(\xi)$. The two independent solutions are take the form [12]

$$\begin{aligned}\mathrm{Ai}(\xi) &= \frac{1}{3}\xi^{1/2}\{I_{-1/3}(\frac{2}{3}\xi^{3/2}) - I_{1/3}(\frac{2}{3}\xi^{3/2})\} \\ &= \frac{\xi^{1/2}}{\pi\sqrt{3}}K_{1/3}(\frac{2}{3}\xi^{3/2}). \qquad (9)\end{aligned}$$

$$\begin{aligned}\mathrm{Bi}(\xi) &= \frac{1}{3}\xi^{1/2}\{I_{-1/3}(\frac{2}{3}\xi^{3/2}) + I_{1/3}(\frac{2}{3}\xi^{3/2})\} \\ &= \frac{2\xi^{1/2}}{\pi\sqrt{3}}\mathcal{K}_{1/3}(\frac{2}{3}\xi^{3/2}). \qquad (10)\end{aligned}$$

We can expand the solutions into the form

$$\mathrm{Ai}(\xi) = c_1 f(\xi) - c_2 g(\xi). \qquad (11)$$

$$\mathrm{Bi}(\xi) = \sqrt{3}\,[c_1 f(\xi) + c_2 g(\xi)]. \qquad (12)$$

where

$$f(\xi) = \sum_{k=0}^{\infty} 3^k \big(\frac{1}{3}\big)_k \frac{\xi^{3k}}{(3k)!}. \qquad (13)$$

$$g(\xi) = \sum_{k=0}^{\infty} 3^k \big(\frac{2}{3}\big)_k \frac{\xi^{3k+1}}{(3k+1)!}. \qquad (14)$$

$$3^k\big(\alpha + \frac{1}{3}\big)_k = (3\alpha+1)(3\alpha+4)\ldots(3\alpha+3k-2), \quad \text{forwhich} \quad k = 1,2,3,\ldots. \qquad (15)$$

with $(\alpha + 1/3)_0 = 1$, $c_1 = 0.355028053887817$ and $c_2 = 0.258819403792807$ [13].

Once again numerical techniques must be used, but it is interesting to use the asymptotic expansion which would take the form

$$\mathrm{Ai}(\xi) \sim \frac{1}{2}\pi^{-1/2}\xi^{-1/4}\exp(-\varsigma)\sum_{k=0}^{\infty}(-1)^k c_k \varsigma^{-k}. \qquad (16)$$

$$\mathrm{Bi}(\xi) \sim \pi^{-1/2}\xi^{-1/4}\exp(-\varsigma)\sum_{k=0}^{\infty} c_k \varsigma^{-k}. \qquad (17)$$

where

$$\varsigma = \frac{2}{3}\xi^{3/2}. \qquad (18)$$

$$c_k = \frac{\Gamma(3k+0.5)}{54^k k!\Gamma(k+0.5)} \qquad (19)$$

of these two functions only $\mathrm{Ai}(\xi)$ are acceptable as wave function because $\mathrm{Bi}(\xi)$ cannot be normalized. Thus the solutions described by Eq. (4) holds. We have then

$$\phi(z) \propto \mathrm{Ai}(\xi) = \mathrm{Ai}(\frac{z}{z_0} - \lambda_n), \quad \lambda_n = \frac{\eta}{z_0}. \qquad (20)$$

with $z_0\lambda_n$ represents the nth classical turning point [10], also described by z_n.

Applying the boundary condition $\phi = 0$ at $z = 0$ then requires $\mathrm{Ai}(-\lambda_n) = 0$, and the energy levels are given by the roots of this equation which is strictly accurate only as $n \to \infty$. This function is an oscillatory function as described in [11] and for large E we may employ the asymptotic form

$$\begin{aligned}\mathrm{Ai}(-x) \sim \pi^{-1/2}x^{-1/4}[&\sin(\varsigma + \frac{\pi}{4})\sum_{k=0}^{\infty}(-1)^k c_{2k}\varsigma^{-2k} \\ &+ \cos(\varsigma + \frac{\pi}{4})\sum_{k=0}^{\infty}(-1)^k c_{2k+1}\varsigma^{-2k-1}]. \qquad (21)\end{aligned}$$

where ς and c_k, $k = 0,1,2,\ldots$ are defined in equation (18) and (19) respectively.

Once again by applying the boundary condition we can reject the cosinus term in Eq. (21) leaving only the sinus term. Thus we have then

$$\mathrm{Ai}(-\lambda_n) \sim \pi^{-1/2}\lambda_n^{-1/4}\sin(\frac{2}{3}\lambda_n^{3/2} + \frac{\pi}{4}). \qquad (22)$$

and obtain $\sin[(2/3)\lambda_n^{3/2} + \pi/4)] \approx 0$, which has solutions

$$\lambda_n = [\frac{3}{2}\pi(n - \frac{1}{4})]^{2/3}. \qquad (23)$$

substituting $\lambda_n = \eta/z_0$ where $\eta = E/mg$ and $z_0 = (\hbar^2/2m^2g)^{1/3}$ we have then

$$E_n = [\frac{9}{8}\pi^2(n - \frac{1}{4})^2 m\hbar^2 g^2]^{1/3}. \qquad (24)$$

This expression, though not quite exact, is always accurate to better than 1% [11].

It can be shown that for large n the levels go like $n^{2/3}$ which may be compared with n^2 for the infinite square well [5]. This is within the result of [14] that the spectrum can rise no faster than n^2 in the nonrelativistic case. The closer spacing is connected with the shallower rate of climb of the potential. Further investigation on this system gives much more understanding about the atomic behaviour.

An experimental demonstration of this problem in the atomic scale presented in [8] of which multiple bouncing of cesium atoms on a reflecting surface was observed. The reflecting surface was made from evanescent wave formed by internal reflection of a quasiresonant laser beam at a curved glass surface. A cold cloud of cesium atoms was dropped onto the mirror and observed to rebound more than 8 times. The earlier demonstration of such reflection with mirror of a reflectivity close to 100% has been achieved and the quantum-state selective reflection of atoms is observed [7].

Following this result was a proposed scheme to create an atomic cavity on the basis of reflection of atoms from a laser field [9] where the main parameters of cavity such as maximum and minimum atomic velocity, cavity stability, scheme of atomic injection, maximum atom density were defined. Later the quantum damping for such system studied in [10]. All the studies can be regarded as a first step towards an interferometer of Fabry-Pérot type

for atomic de Broglie waves [9, 15].

The author would like to thank M. M. Nieto for comment on the previous edition of this note.

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